University of Florida

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Department of Mathematics

College *of* Liberal Arts *and* Sciences

MAS 7216 and MAT 6932 and MAT 4930 – Number Theory – Irrationality, Diophantine Approximations and Transcendence

COURSE SYLLABUS

INSTRUCTOR: Krishnaswami Alladi

DATES & TIME: MWF 6-th period (12:50 – 1:40 pm)

ROOM: ZOOM

ZOOM OFFICE HOURS: M and W 7th period and by appointment

COURSE DESCRIPTION: The study of irrational numbers dates back to Greek antiquity. Yet the subject remains an active area of research today. Although "almost all" real numbers are irrational, it is very difficult to establish the irrationality of a given number. We will systematically study a variety of techniques which will help in confirming the irrationality of different classes of numbers starting with Dirichlet's fundamental criterion for irrationality. Following this, we will discuss irrationality criteria utilizing series and product representations of reals due to Engel, Cantor and Sylvester, regular and general continued fraction expansions, and Farey fractions. We will present various proofs of the irrationality of important numbers such as e and pi, and the irrationality of functions like the trigonometric, hyperbolic and the Bessel functions at rational arguments. Next we will take up the study of the closeness of approximation of irrationals by rationals and introduce the concept of irrationality measures. This is the subject of Diophantine approximations.

It will be shown that the truncations of the regular continued fraction expansion of real numbers generate the sequence of "best approximations", but it is very difficult to determine the continued fraction expansion of a given irrational.

We will develop methods to obtain efficient irrationality measures. This will involve the use of Legendre polynomials and Pade approximations. An important outcome of this approach is the irrationality of the Riemann zeta function at the odd integer 3 – a fact long conjectured but only established in 1978. The study of irrationality measures also explains why Pell's equation

such as $x^2 – 2y^2 = 1$ has infinitely many solutions whereas the Thue equation: $x^3 – 2y^3 = 1$ has only finitely many integer solutions. Thus the subject of Diophantine approximations is closely associated with the theory of Diophantine equations. As part of our study of Diophantine approximations, we will include a discussion of uniform distribution and of normal numbers – numbers whose decimal digits are statistically uniformly distributed. Finally, we will launch the study of transcendental numbers by showing first that e is transcendental, and more generally that the exponential function takes transcendental values at non-zero algebraic arguments. Since $\exp(i\pi)=-1$ this result of Lindemann at the end of the 19-th century implies that π is transcendental and thereby shows the impossibility of "squaring the circle" – one of three problems of Greek antiquity. We also hope to discuss measures of transcendence and the transcendence of some fascinating functions (defined by lacunary series) at algebraic arguments.

A host of great problems remain unsolved **:** Is Euler's constant γ irrational?

Are the values of the Riemann zeta function at the odd integers >3 irrational?

What is the precise irrationality measure for π? We know that "almost all" numbers are normal to all bases, but we do not know a single example of a number whose digit expansions to all integer bases (not just base 10) are normal.

The course will be completely self contained and can be followed by any of our graduate students and those in allied disciplines as well (such as computer science and statistics). I will use my own detailed notes for the course but will give a number of books as references.

The list of topics given above is vast, and I will cover as many of them as time permits and depending on the interest of the audience. My goal is not to race through the topics, but do a thorough discussion at a pace that is comfortable. If there is interest, this course might be extended into Spring 2021 as MAS 7216.

TEXT: There is no prescribed text for the course. I have elaborate Notes which I will use. References to various texts will be given.

GRADING: Grades will be based on homework assigned.

DIVERSITY STATEMENT: I am committed to diversity and inclusion of all students in this course. I acknowledge, respect, and value the diverse nature, background and perspective of students and believe that it furthers academic achievements. It is my intent to present materials and activities that are respectful of diversity: race, color, creed, gender, gender identity, sexual orientation, age, religious status, national origin, ethnicity, disability, socioeconomic status, and any other distinguishing qualities.

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assignment." The Honor Code specifies a number of behaviors that are in violation of this code and the possible sanctions. Furthermore, you are obligated to report any condition that facilitates academic misconduct to appropriate personnel. If you have any questions or concerns, please consult with the instructor in this class."

Class Attendance: "Requirements for class attendance and make-up exams, assignments, and other work in this course are consistent with university policies that can be found at: https://catalog.ufl.edu/ugrad/current/regulations/info/attendance.aspx ."

Accommodations for Students with Disabilities: "Students with disabilities who experience learning barriers and would like to request academic accommodations should connect with the disability Resource Center by visiting https://disability.ufl.edu/students/get-started/. It is important for students to share their accommodation letter with their instructor and discuss their access needs, as early as possible in the semester."

Online Evaluations: "Students are expected to provide professional and respectful feedback on the quality of instruction in this course by completing course evaluations online via GatorEvals. Guidance on how to give feedback in a professional and respectful manner is available at https://gatorevals.aa.ufl.edu/students/ . Students will be notified when the evaluation period opens, and can complete evaluations through the email they receive from GatorEvals, in their Canvas course menu under GatorEvals, or via https://ufl.bluera.com/ufl/ . Summaries of course evaluation results are available to students at https://gatorevals.aa.ufl.edu/public-results/ ."

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